Quantum-hydrodynamical picture of the massive Higgs boson

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Abstract. The phenomenon of spontaneous symmetry breaking admits a physical interpretation in terms of the Bose condensation process of elementary spinless quanta. In this picture, the broken-symmetry phase emerges as a real physical medium, endowed with a hierarchical pattern of scales, supporting two types of elementary excitations for $\mathbf{k} \to 0$: a massive energy branch $E_a(\mathbf{k}) \to M_H$, corresponding to the usual Higgs boson field, and a collective gapless branch $E_b(\mathbf{k}) \to 0$. This is similar to the coexistence of phonons and rotons in superfluid ^{4}He that, in fact, is usually considered the condensed-matter analog of the Higgs condensate. After previous work dedicated to the properties of the gapless phonon branch, in this paper we use quantum hydrodynamics to propose a physical interpretation of the massive branch. On the base of our results, M_H coincides with the energy gap for vortex formation and a massive Higgs boson is like a roton in superfluid ^{4}He. Within this interpretation of the Higgs particle, there is no naturalness problem since M_H remains a naturally intermediate, fixed energy scale, even for an ultimate ultraviolet cutoff $\Lambda \to \infty$.

1 Introduction

The idea of a spontaneously broken phase as a "condensate" is now widely accepted. For instance, in the physically relevant case of the standard model, the situation can be summarized saying [1] that "What we experience as empty space is nothing but the configuration of the Higgs field that has the lowest possible energy. If we move from field jargon to particle jargon, this means that empty space is actually filled with Higgs particles. They have Bose condensed."

Here, by "Bose condensation" one means the phenomenon of spontaneous particle creation in the same quantum state ($\mathbf{k} = 0$ in some reference frame) from the empty vacuum $|0\rangle$ of perturbation theory. In this way, the translation from "field jargon to particle jargon", amounts to establish a well-defined functional relation (see [2] and Sect. 2) $n = n(\phi^2)$ between the average density of scalar quanta, the "phions", and the average value of the scalar field $\langle \Phi \rangle = \phi$. Thus, Bose condensation is just a consequence of the minimization condition of the effective potential $V_{\mathrm{eff}}(\phi)$. This has absolute minima at some values $\phi = \pm v \neq 0$ for which $n(v^2) = \bar{n} \neq 0$ [2].

Of course, in order for this picture to be consistent, spontaneous symmetry breaking should occur for physical (i.e. real and non-negative) values of the phion mass m_Φ. In other words, Bose condensation requires the phase transition in (cutoff) $\lambda \Phi^4$ theories to be *first order*. While in the presence of gauge bosons this can be shown perturbatively [3], the use of perturbation theory in a pure $\lambda \Phi^4$ theory leads to contradictory results between even and odd orders [4]. Therefore, one has to analyze the effective potential beyond perturbation theory. By relying on the assumed exact "triviality" property of the theory in $3 + 1$ space-time dimensions [5], one is driven to consider the class of "triviality compatible" approximations [2, 6, 7] to the effective potential, say $V_{\mathrm{eff}}(\phi) = V_{\mathrm{triv}}(\phi)$. This includes the one-loop potential, the gaussian and post-gaussian [8] calculations of the Cornwall–Jackiw–Tomboulis [9] effective potential for composite operators, i.e. all approximations where the fluctuation field is governed by an effective quadratic Hamiltonian. In all such cases, the lowest energy state of the massless theory at $m_\Phi = 0$ corresponds to a broken-symmetry phase as first shown in [10] for the gaussian approximation. Therefore the phase transition, occurring earlier for small positive values of m_Φ, is first order. However, there is a subtlety since it corresponds to an *infinitesimally weak* first-order phase transition. In fact, it is first order for any finite ultraviolet cutoff Λ but becomes asymptotically second order[1] in the continuum limit $\Lambda \to \infty$.

In any case, for all finite values of Λ, a *particle-gas* picture of the underlying scalar system is possible. This can be formulated in terms of two basic quantities: the equilibrium phion density $\bar{n}$ and the phion–phion scattering length a,

$$a = \frac{\lambda}{8\pi m_\Phi}, \tag{1}$$

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[1] In the presence of gauge bosons, e.g. in scalar electrodynamics, $V_{\mathrm{triv}}(\phi)$ predicts a first-order transition as well, consistently with the perturbative Coleman–Weinberg analysis; see [11]

defined, in the limit of zero-momentum scattering, from the dimensionless scalar self-coupling λ and the phion mass.

As shown in [2], these two quantities combine to produce all relevant length scales of the system,

$$a \ll \frac{1}{\sqrt{\bar{n}a}} \ll \frac{1}{\bar{n}a^2}, \qquad (2)$$

that decouple for an infinitely dilute system where $\bar{n}a^3 \to 0$. In this situation, which corresponds to approaching the continuum limit of quantum field theory [2], one discovers an unexpected result: the *hierarchical* nature of the scalar condensate.

At the same time, a particle-gas picture of the broken-symmetry phase raises several questions. For instance, in condensed media the properties of the system over vastly different scales are described by different branches of the energy spectrum. Are there similar transitions in the scalar condensate? Also, what about the coexistence of *exact* Lorentz covariance and vacuum condensation in *effective* quantum field theories? Can the violations of locality, at the energy scale fixed by the ultraviolet cutoff, induce non-Lorentz covariant modifications of the *infrared* energy spectrum that depends on the vacuum structure [12]?

To indicate this type of infrared–ultraviolet connection, originating from vacuum condensation in effective quantum field theories, Volovik [13] has introduced a very appropriate name: re-entrant violations of special relativity in the low energy corner. These occur in a small shell of three-momenta, say $|\mathbf{k}| < \delta$, that only vanishes in the strict local limit where $\Lambda \to \infty$ and an exact Lorentz covariant energy spectrum is re-obtained in the whole range of momenta.

By denoting M_H as the typical energy scale associated with the Lorentz covariant part of the energy spectrum, the "re-entrant" nature of the violating effects means that $\mathcal{O}\left(\frac{\delta}{M_H}\right)$ vacuum-dependent corrections are equivalent to $\mathcal{O}\left(\frac{M_H}{\Lambda}\right)$ effects (see below). The $1/\Lambda$ terms, that have always been neglected when discussing [14] how Lorentz covariance emerges in effective theories when removing the ultraviolet cutoff, although infinitesimally small, can play an important role over distances larger than Λ/M_H^2, i.e. infinitely larger than the typical elementary particle scale $\xi_H = 1/M_H$.

As discussed in [15–17], the basic ingredient to understand the nature of the "re-entrant" effects in the scalar condensate consists in a purely quantum field-theoretical result: the *two-valued* nature of the connected zero-four-momentum propagator $G^{-1}(k = 0)$ in the broken phase [18,19]. In fact, besides the well-known massive solution $G_a^{-1}(k = 0) = M_H^2$, one also finds $G_b^{-1}(k = 0) = 0$.

The b-type of solution corresponds to processes where absorbing (or emitting) a very small three-momentum $\mathbf{k} \to 0$ does not cost any finite energy. This situation is well known in a condensed medium, where a small three-momentum can be coherently distributed among a large number of elementary constituents, and corresponds to the hydrodynamical regime of density fluctuations whose wavelengths $2\pi/|\mathbf{k}|$ are *larger* than r_{mfp}, the mean free path for the elementary constituents.

In this sense, the situation is very similar to superfluid ^{4}He, where the observed energy spectrum is due to the peculiar transition from the "phonon branch" to the "roton branch" at a momentum scale $|\mathbf{k}_0|$ where $E_{\mathrm{phonon}}(\mathbf{k}_0) \sim E_{\mathrm{roton}}(\mathbf{k}_0)$. The analog for the scalar condensate amounts to an energy spectrum with the following limiting behaviors:
(i) $E(\mathbf{k}) \to E_b(\mathbf{k}) \sim c_{\mathrm{s}}|\mathbf{k}|$ for $\mathbf{k} \to 0$; (ii) $E(\mathbf{k}) \to E_a(\mathbf{k}) \sim M_H + \frac{\mathbf{k}^2}{2M_H}$ for $|\mathbf{k}| \gtrsim \delta$, where the characteristic momentum scale $\delta \ll M_H$, at which $E_a(\delta) \sim E_b(\delta)$, marks the transition from collective to single-particle excitations. This occurs for

$$\delta \sim 1/r_{\mathrm{mfp}}, \qquad (3)$$

where [20,21]

$$r_{\mathrm{mfp}} \sim \frac{1}{\bar{n}a^2} \qquad (4)$$

is the phion mean free path, for a given value of the phion density $n = \bar{n}$ and a given value of the phion–phion scattering length a. In terms of the same quantities, one also finds (see [2] and Sect. 2)

$$M_H^2 \sim \bar{n}a, \qquad (5)$$

giving the anticipated trend of the dimensionless ratios ($\Lambda \sim 1/a$)

$$\frac{\delta}{M_H} \sim \frac{M_H}{\Lambda} \sim \sqrt{\bar{n}a^3} \to 0, \qquad (6)$$

in the continuum limit where $a \to 0$ and the mass scale $\bar{n}a$ is held fixed[2].

Now, deducing the detailed form of the energy spectrum that interpolates between $E_a(\mathbf{k})$ and $E_b(\mathbf{k})$ is a formidable task. To have an idea, the same problem in superfluid ^{4}He, after more than fifty years and despite the efforts of many theorists, notably Landau and Feynman, has not been solved in a satisfactory way. Therefore, by taking into account the above remark, one can simply approximate [15–17] the expansion of the scalar field in the broken phase by two separate branches as (phys is for "physical")[3]

$$\Phi_{\mathrm{phys}}(x) = v_R + h(x) + H(x), \qquad (7)$$

with

$$h(x) = \sum_{|\mathbf{k}|<\delta} \frac{1}{\sqrt{2VE_k}} \left[\tilde{h}_{\mathbf{k}} e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-E_k t)} + (\tilde{h}_{\mathbf{k}})^\dagger e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-E_k t)} \right], \qquad (8)$$

and

$$H(x) = \sum_{|\mathbf{k}|>\delta} \frac{1}{\sqrt{2VE_k}} \left[\tilde{H}_{\mathbf{k}} e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-E_k t)} + (\tilde{H}_{\mathbf{k}})^\dagger e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-E_k t)} \right], \qquad (9)$$

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[2] In connection with the relation (6) $M_H^2 \sim \delta\Lambda$, notice the formal analogy with models [22] containing extra space-time dimensions with a "large" compactification size $R_c = \mathcal{O}(1)$ mm. In this case, in fact, by introducing the electroweak scale M_{EW}, the Planck scale M_{Planck} and $M_c \equiv 1/R_c$, one gets $M_{\mathrm{EW}}^2 \sim M_c M_{\mathrm{Planck}}$

[3] By "physical" we mean that the normalization of the vacuum field v_R is such that the quadratic shape of $V_{\mathrm{eff}}(\phi_R)$ at $\phi_R = v_R$ is precisely given by the physical Higgs mass M_H^2

where V is the quantization volume and $E_k = c_s|\mathbf{k}|$ for $|\mathbf{k}| < \delta$ while $E_k = \sqrt{\mathbf{k}^2 + M_H^2}$ for $|\mathbf{k}| > \delta$. Also, $c_s\delta \sim M_H$.

Equations (7)–(9) replace the more conventional relations

$$\Phi_{\mathrm{phys}}(x) = v_R + H(x), \qquad (10)$$

where

$$H(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\mathsf{V}E_k}} \left[\tilde{H}_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{x} - E_k t)} + (\tilde{H}_{\mathbf{k}})^\dagger e^{-i(\mathbf{k}\cdot\mathbf{x} - E_k t)} \right], \qquad (11)$$

with $E_k = \sqrt{\mathbf{k}^2 + M_H^2}$. Equations (10) and (11) are re-obtained in the limit $\frac{\delta}{M_H} \sim \frac{M_H}{\Lambda} \to 0$ where the wavelengths associated to $h(x)$ become infinitely large in units of the physical scale set by $\xi_H = 1/M_H$. In this limit, where for any finite value of $\mathbf{k}$ the broken phase has only massive excitations, one recovers an exactly Lorentz covariant theory.

Now, as anticipated, the interpretation of the gapless branch in terms of density fluctuations places no particular problem [15–17]. In fact, "Any quantum liquid consisting of particles with integral spin (such as the liquid isotope ^{4}He) must certainly have a spectrum of this type... In a quantum Bose liquid, elementary excitations with small momenta $\mathbf{k}$ (wavelengths large compared with distances between atoms) correspond to ordinary hydrodynamic sound waves, i.e. they are phonons. This means that the energy of such quasi-particles is a linear function of their momentum" [23]. In this sense, a superfluid vacuum provides for $\mathbf{k} \to 0$ a universal picture. This result does not depend on the details of the short-distance interaction and even on the nature of the elementary constituents. For instance, the same coarse-grained description is found in superfluid fermionic vacua [24] that, as compared to the Higgs vacuum, bear the same relation as superfluid ^{3}He has to superfluid ^{4}He.

On the other hand, a full analogy with ^{4}He would also require one to establish the interpretation of the massive branch for $|\mathbf{k}| \gtrsim \delta$ as a "roton", i.e. in terms of a suitable vortical motion in the superfluid, and this is by no means obvious. For instance, Landau's roton spectrum

$$E_{\mathrm{roton}}(\mathbf{k}) = \Delta + \frac{\mathbf{k}^2}{2\mu}, \qquad (12)$$

depends on two parameters Δ and μ that, in superfluid ^{4}He, are vastly different. In fact (in units $\hbar = c = 1$, where Δ and μ have the same physical dimensions), one finds $\Delta \sim 7 \cdot 10^{-4}$ eV and $\mu \sim 6 \cdot 10^8$ eV while $E_a(\mathbf{k}) = \sqrt{\mathbf{k}^2 + M_H^2}$ depends on a single mass parameter M_H. Under which conditions can a superfluid medium exhibit rotons with $\Delta = \mu$? Moreover, even if $\Delta = \mu$, does this value agree with the Higgs mass parameter M_H obtained in quantum field theory?

The answer to this type of questions can only be obtained by combining a field-theoretical description of the condensation phenomenon with the basic ingredients of quantum hydrodynamics. This analysis represents the main content of this paper and will be presented in the following. In Sect. 2, we shall first review the formalism of [2] with the hierarchical pattern of scales that is established in the scalar condensate. Further, in Sect. 3, we shall use the formalism of quantum hydrodynamics and discuss the interpretation of the massive branch as a roton. Finally, Sect. 4 will contain a summary together with other possible consequences of our approach.

2 The Higgs condensate and a hierarchy of scales

We shall now first resume the main results of [2] in the case of a one-component $\lambda\Phi^4$ theory, a system where the condensing quanta are just neutral spinless particles, the "phions".

One starts by quantizing the scalar field $\Phi(\mathbf{x})$ in terms of $a_{\mathbf{k}}$, $a_{\mathbf{k}}^\dagger$, the annihilation and creation operators for the elementary phions whose "empty" vacuum state $|0\rangle$ is defined through $a_{\mathbf{k}}|0\rangle = \langle 0|a_{\mathbf{k}}^\dagger = 0$.

The phion system is assumed to be contained within a finite box of volume V with periodic boundary conditions. There is then a discrete set of allowed modes $\mathbf{k}$. In the end one takes the infinite-volume limit and the summation over allowed modes goes over to an integration: $\sum_{\mathbf{k}} \to \mathsf{V} \int d^3k/(2\pi)^3$. In this way, the scalar field is expressed as

$$\Phi(\mathbf{x}) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\mathsf{V}E_k}} \left[a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \qquad (13)$$

where $E_k = \sqrt{\mathbf{k}^2 + m_\Phi^2}$, m_Φ being the physical, renormalized phion mass.

Bose condensation means that in the ground state there is an average number N_0 of phions in the $\mathbf{k} = 0$ mode, where N_0 is a finite fraction of the total average number N:

$$\mathsf{N} = \left\langle \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \right\rangle. \qquad (14)$$

At zero temperature, if the gas is dilute, almost all the particles are in the condensate; $\mathsf{N}_0(T=0) \sim \mathsf{N}$. In fact, the fraction which is not in the condensate ("depletion"),

$$D = 1 - \frac{\mathsf{N}_0}{\mathsf{N}} = \mathcal{O}(\sqrt{na^3}), \qquad (15)$$

is a phase-space effect that becomes negligible for a very dilute system [25] where

$$\epsilon = \sqrt{na^3} \ll 1. \qquad (16)$$

In (15) and (16) we have introduced the phion density

$$n = \frac{\mathsf{N}}{\mathsf{V}}, \qquad (17)$$

and the phion–phion scattering length (1).

Therefore, for a very dilute system, where, to a first approximation, one neglects the residual operator part of $a_{\mathbf{k}=0}$, one gets $a_{\mathbf{k}=0}^{\dagger} a_{\mathbf{k}=0} \sim \mathsf{N}$ and so, $a_{\mathbf{k}=0}$ can be identified with the c-number $\sqrt{\mathsf{N}}$ (up to a phase factor). In this way

$$\phi = \langle \Phi \rangle = \frac{1}{\sqrt{2 \mathsf{V} m_{\Phi}}} \langle (a_{\mathbf{k}=0}^{\dagger} + a_{\mathbf{k}=0}) \rangle \sim \sqrt{\frac{2\mathsf{N}}{\mathsf{V} m_{\Phi}}}, \quad (18)$$

or

$$n(\phi^2) \sim \tfrac{1}{2} m_{\Phi} \phi^2. \quad (19)$$

With this identification, setting $a_{\mathbf{k}=0}^{\dagger} = a_{\mathbf{k}=0} = \sqrt{\mathsf{N}}$ is equivalent to shifting the quantum field Φ by a constant term ϕ. Further, using (19), the energy density $\mathcal{E} = \mathcal{E}(n)$ can be translated into the effective potential

$$V_{\mathrm{eff}}(\phi) = \mathcal{E}(n). \quad (20)$$

Now, by exploring the limit $m_{\Phi} \to 0$ in the class of "triviality compatible" approximations to the effective potential [2,6,7] one discovers non-trivial absolute minima $\phi = \pm v \neq 0$ of $V_{\mathrm{eff}}(\phi) = V_{\mathrm{triv}}(\phi)$ and Bose condensation with an average density $\bar{n} = n(v^2)$.

The basic relations of the broken phase are

$$M_H^2 \sim \lambda v^2 \sim \bar{n} a, \quad (21)$$

and

$$m_{\Phi}^2 \sim \lambda^2 v^2 \sim \epsilon^2 \bar{n} a. \quad (22)$$

The key ingredient in understanding why the phase transition occurs for a value of m_{Φ} that is still positive consists in the observation [2] that the phion–phion interaction is not always repulsive. Besides the contact $+\lambda \delta^{(3)}(\mathbf{r})$ potential there is an induced attraction $-\lambda^2 \frac{e^{-2m_{\Phi} r}}{r^3}$ from ultraviolet-finite parts of higher order graphs (see also [26]) that, differently from the usual ultraviolet divergences, cannot be re-absorbed into a standard redefinition of the tree-level coupling. For small values of λ and for sufficiently small values of m_{Φ} the corresponding graphs, when taken into account consistently in the effective potential, can compensate for the effects of both the short range repulsion and of the non-zero phion mass. In this situation, the perturbative empty vacuum state $|0\rangle$, although locally stable, is not globally stable and the lowest energy state becomes a state with a non-zero density of phions that are Bose condensed in the zero-momentum state.

We emphasize that this weakly first-order scenario of symmetry breaking is discovered in a *class* of approximations to the effective potential: just those that are consistent with the assumed exact "triviality" property of the theory in $3 + 1$ space-time dimensions [5]. In any case, it can be objectively tested against the standard picture based on a second-order phase transition. To this end one can run numerical simulations near the phase transition region and compare the predictions of [6,7] with the conventional existing two-loop or renormalization-group-improved forms of the effective potential. When this is done, the quality of the fits to the existing lattice data [27,28] favors unambiguously the first-order scenario of [2,6,7].

Let us now consider the range of momenta associated with the two different branches of the energy spectrum. In condensed matter, the transition between the acoustic branch and the single-particle spectrum corresponds to their matching at a momentum scale set by the inverse mean free path for the elementary constituents. As anticipated in the Introduction, in the scalar condensate the matching condition corresponds to a momentum scale $\delta \ll M_H$ where $E_a(\delta) \sim E_b(\delta)$ or

$$c_{\mathrm{s}} \delta \sim \sqrt{\delta^2 + M_H^2} \sim M_H + \frac{\delta^2}{2M_H}, \quad (23)$$

with $\delta \sim \frac{1}{r_{\mathrm{mfp}}}$, r_{mfp} being the phion mean free path for $n = \bar{n}$ in (4).

Now, the scattering length a can be used to define a far ultraviolet scale:

$$\Lambda \equiv 1/a. \quad (24)$$

up to which phions can be treated as "hard spheres". Using (3), (4), (21) and (23), this yields

$$t = \frac{\Lambda}{M_H} \sim \sqrt{\frac{1}{\bar{n} a^3}}, \quad (25)$$

and

$$\frac{1}{c_{\mathrm{s}}} \sim \frac{\delta}{M_H} \sim \sqrt{\bar{n} a^3}. \quad (26)$$

Therefore, in the continuum limit where $t \to \infty$ one gets an infinitely dilute Higgs condensate where $\epsilon = \sqrt{\bar{n} a^3} \to 0$ and the hierarchy of scales

$$\delta \ll M_H \ll \Lambda, \quad (27)$$

is related as in (6).

Finally, by using (1), the condition for spontaneous symmetry breaking (22) can also be expressed as

$$m_{\Phi} \sim \bar{n} a^2 \sim \frac{1}{r_{\mathrm{mfp}}}, \quad (28)$$

so that the phion mean free path in the condensate is of the same order as the phion Compton wavelength $1/m_{\Phi}$.

Notice that the order of magnitude of the sound velocity,

$$c_{\mathrm{s}} \sim \frac{M_H}{\delta} \sim \frac{\Lambda}{M_H} \sim \frac{1}{\epsilon}, \quad (29)$$

is much larger than unity (in units of the light velocity $c = 1$). Actually, c_{s} *must* diverge in the continuum limit where Lorentz covariance becomes exact [15]. In this limit, where the vacuum acquires an infinite rigidity, the condensate becomes incompressible and the massive branch of the energy spectrum remains valid down to $\mathbf{k} = 0$.

The presence of superluminal sound in the scalar condensate has different motivations. First of all we observe that, on general grounds, "... it is an open question whether $\frac{c_{\mathrm{s}}}{c}$ remains less than unity when non-electromagnetic forces

are taken into account" [29]. For this reason, several authors [30–32] have considered the possibility of media whose long-wavelength compressional modes for $\mathbf{k} \rightarrow 0$ have phase and group velocity $\frac{E}{|\mathbf{k}|} = \frac{dE}{d|\mathbf{k}|} = c_s > c$.

This possibility depends on the approximate nature of locality in cutoff-dependent quantum field theories where the elementary quanta are treated as "hard spheres". In this case, in fact, a hard-sphere radius is known [33] to imply a superluminal propagation within the sphere boundary. Now, in the perturbative empty vacuum state (with no condensed quanta) such superluminal propagation is restricted to very short wavelengths, smaller than the inverse ultraviolet cutoff. However, in the condensed vacuum, the hard spheres can "touch" each other so that the actual propagation of density fluctuations in a hard-sphere system might take place at a superluminal speed. This intuitive idea is at the base of the "macroscopic" violations of locality discussed in [30] (and of the "re-entrant" violations of special relativity in the low energy corner [13] mentioned above).

In some cases, superluminal sound is known to arise [31] when a large negative bare mass and a large positive self-energy combine to produce a very small physical mass, just the situation expected for the quanta of the scalar condensate. In this way, the physical origin of the superluminal sound is traced back to the asymmetric role of mass renormalization: it subtracts out self-interaction energy without altering the tree-level interparticle interactions that contribute to the pressure.

On the other hand, following [15, 21], superluminal sound is also consistent with the equation of state of a perfect fluid whose energy density has a minimum at some given value of the particle density, as it happens in the scalar condensate. Just for this reason, near the minimum, long-wavelength density fluctuations represent nearly instantaneous effects that can propagate at arbitrarily large speed. In this sense, the scalar condensate rensembles an elastic medium near the incompressibility limit where the Poisson ratio $\nu \rightarrow 1/2$ and the propagation speed of the longitudinal waves of dilatation diverges in units of the propagation speed of the transverse waves of distortion [16].

Summarizing the previous results, we find that in the local limit of the theory, where $\Lambda/M_H \rightarrow \infty$, one also finds $\delta/M_H \rightarrow 0$ so that the energy spectrum $E(\mathbf{k})$ reduces to $E_a(\mathbf{k}) = \sqrt{\mathbf{k}^2 + M_H^2}$ in the whole range of $|\mathbf{k}|$. In the cutoff theory, however, one should expect infinitesimal deviations in an infinitesimal region of three-momenta. For instance, assuming $\Lambda = 10^{19}\,\text{GeV}$ and $M_H = 250\,\text{GeV}$, a scale $\delta \sim \frac{M_H^2}{\Lambda} \sim 10^{-5}\,\text{eV}$, for which $\frac{\delta}{M_H} \sim 4 \cdot 10^{-17}$, might well represent the physical realization of a formally infinitesimal quantity. If this were the right order of magnitude, the collective density fluctuations of the Higgs vacuum described by $E_b(\mathbf{k})$ have wavelengths $> \frac{2\pi}{\delta}$, thus ranging from about a centimeter up to infinity [15–17].

On the other hand, for $|\mathbf{k}| \gtrsim \delta$, the excitation spectrum describes single-particle states of mass $M_H \sim \sqrt{\bar{n}a}$. In the following section, we shall show that these states can be interpreted as elementary excitations associated with a vortical motion.

3 The "roton" picture of the massive branch

In his theory, Landau suggested that in a superfluid medium there must be elementary vortex excitations, the "rotons", whose energy has the form of (12). In his original papers [34], Landau did not work out an explicit derivation of (12). This was, however, deduced subsequently by Ziman [35] whose formalism we shall briefly resume in the following (for the convenience of the reader we shall adopt in this section the same notation as [35]).

Ziman's starting point is the form of the Hamiltonian density of a fluid,

$$H = \frac{1}{2}\rho \mathbf{u}^2 + \rho W(\rho), \qquad (30)$$

where ρ is the mass density and W the internal energy whose mimimum is obtained for $\rho \equiv \bar{\rho}$. The fluid velocity

$$\mathbf{u} = -\nabla\varphi - \frac{\mathrm{i}}{2\rho}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*), \qquad (31)$$

is expressed in terms of the three Clebsch potentials [36–38] φ, Ψ and Ψ^*. Notice that, although the fluid is described in terms of the four-component field $(\rho, \varphi, \Psi, \Psi^*)$, there are no fundamental "charges" and all dynamical effects derive from the possible states of motion of the fluid.

In quantum hydrodynamics (ρ, φ) and (Ψ, Ψ^*) are pairs of canonically conjugated variables, i.e.

$$[\rho(\mathbf{r}), \varphi(\mathbf{r}')] = \mathrm{i}\delta^3(\mathbf{r} - \mathbf{r}'), \qquad (32)$$

and

$$[\Psi(\mathbf{r}), \Psi^*(\mathbf{r}')] = \delta^3(\mathbf{r} - \mathbf{r}'), \qquad (33)$$

all other commutators being zero. In this way, one obtains Landau's relations for the commutators of the velocity components:

$$[\mathbf{u}_x(\mathbf{r}), \mathbf{u}_y(\mathbf{r}')] = \frac{1}{\mathrm{i}\rho}\delta^3(\mathbf{r} - \mathbf{r}')\zeta_z, \qquad (34)$$

where

$$\zeta = \nabla \times \mathbf{u} \qquad (35)$$

is the vorticity.

In the incompressibility limit, where $\rho = \bar{\rho}$ and the phase φ are constant throughout the volume of the fluid, the fluid Hamiltonian density reduces to its "roton" part

$$H_{\text{roton}} = -\frac{1}{8\bar{\rho}}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)^2. \qquad (36)$$

In a cubic box of volume V, one can expand in plane waves

$$\Psi = \frac{1}{\sqrt{\mathsf{V}}}\sum_{\mathbf{k}} b_{\mathbf{k}}e^{+\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \quad \Psi^* = \frac{1}{\sqrt{\mathsf{V}}}\sum_{\mathbf{k}} b_{\mathbf{k}}^*e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \qquad (37)$$

with

$$[b_{\mathbf{k}}, b_{\mathbf{l}}^*] = \delta_{\mathbf{k}\mathbf{l}}, \qquad (38)$$

so that by integrating the Hamiltonian density over the whole volume one gets quadratic $b_{\mathbf{k}}^* b_{\mathbf{k}}$ terms. These can be used to define a free-roton Hamiltonian

$$\mathcal{H}_{\text{roton}}^{(0)} = \frac{1}{8\bar{\rho}\mathsf{V}} \sum_{\mathbf{kl}} (\mathbf{k}^2 + \mathbf{l}^2) b_{\mathbf{k}}^* b_{\mathbf{k}}, \qquad (39)$$

that, after converting the discrete sum into an integral, gives finally

$$E_{\text{roton}}(\mathbf{k}) = \frac{1}{8\bar{\rho}} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} (\mathbf{k}^2 + \mathbf{q}^2). \qquad (40)$$

Equation (40) is formally divergent, so that, to extract the relevant values of Δ and μ in (12), one has to find a suitable cutoff momentum for the single-roton excitations, say $|\mathbf{q}| = q_{\text{max}}$. If this is done, we obtain

$$\Delta = \frac{q_{\text{max}}^5}{80\pi^2 \bar{\rho}} \qquad (41)$$

and

$$\mu = \frac{24\pi^2 \bar{\rho}}{q_{\text{max}}^3}. \qquad (42)$$

As a consequence of the introduction of q_{max}, the uncertainty associated with any cutoff procedure will only allow an order of magnitude estimate to be made of Δ and μ and of the regime of parameters associated with the Lorentz covariant condition $\Delta = \mu^4$.

To obtain an estimate of q_{max}, we observe that using the operators $b_{\mathbf{k}}^*$ and $b_{\mathbf{k}}$ one can construct the roton Fock space labelled by $N_{\mathbf{k}}$, the eigenvalue of the roton number operator $b_{\mathbf{k}}^* b_{\mathbf{k}}$, the single-roton states corresponding to $N_{\mathbf{k}} = 1$. Now, by computing the value of the vorticity vector in a single-roton state, one gets the idea of the roton "... as a steady rotational motion of the fluid, capable of moving as a "vortex" through the liquid" [35].

This observation arises by inspection of the single-roton wave functions expressed in cylindrical co-ordinates (r, θ, z). In this case, by introducing the squared transverse momentum $\kappa^2 = k_x^2 + k_y^2$, the azimuthal quantum number ν, and the z-component of the momentum k_z, one finds the equivalent form for the free-roton Hamiltonian

$$\mathcal{H}_{\text{roton}}^{(0)} \qquad (43)$$
$$= \frac{1}{8\bar{\rho}\mathsf{V}} \sum_{\kappa,\kappa',\nu,\nu',k_z,k_z'} (\kappa^2 + k_z^2 + (\kappa')^2 + (k_z')^2) b_{\kappa\nu k_z}^* b_{\kappa\nu k_z},$$

whose single-particle eigenvalues

$$E_{\kappa\nu k_z} = \Delta + \frac{\kappa^2 + k_z^2}{2\mu} \qquad (44)$$

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[4] An effective cutoff in momentum space has been widely used in the literature before the introduction of the idea of renormalization. For instance Bethe's [39] and Welton's [40] treatments of the Lamb shift, both based on the choice $q_{\text{max}} = m_e c$, capture the essential physics and the exact order of magnitude of the effect although it is hard to render them into a fully satisfactory calculation.

have the same Δ and μ as in (41) and (42) with $q_{\text{max}}^2 = (\kappa^2 + k_z^2)_{\text{max}}$. This can be understood by first noticing that the quantities Δ and μ can be expressed in terms of the number of quantum states with $|\mathbf{q}| \leq q_{\text{max}}$, say $g(q_{\text{max}})$, as

$$\Delta \sim q_{\text{max}}^2 \frac{g(q_{\text{max}})}{\bar{\rho}\mathsf{V}} \qquad (45)$$

and

$$\mu \sim \frac{\bar{\rho}\mathsf{V}}{g(q_{\text{max}})}. \qquad (46)$$

Further, using the results of [41], one finds the same leading behavior

$$\frac{g(q_{\text{max}})}{\mathsf{V}} \sim \frac{q_{\text{max}}^3}{6\pi} \qquad (47)$$

by switching from cubical to cylindrical geometry. Finally, averaging over all possible orientations gives $(k_x^2)_{\text{max}} = (k_y^2)_{\text{max}} = (k_z^2)_{\text{max}}$ and one obtains $q_{\text{max}}^2 \sim \frac{3}{2}(\kappa^2)_{\text{max}}$.

In cylindrical coordinates, the single-particle wave functions can be expressed as

$$\psi = \psi_{\kappa,\nu,k_z}(r, \theta, z) = \mathcal{N} J_\nu(\kappa r) \mathrm{e}^{\mathrm{i}\nu\theta} \mathrm{e}^{\mathrm{i}k_z z}, \qquad (48)$$

where $J_\nu(\kappa r)$ are Bessel functions and $\mathcal{N}$ is a normalization factor. In this case, one finds $u_r = u_z = 0$ with the only non-vanishing component of the velocity being

$$u_\theta = \frac{1}{2\mathrm{i}r} \left(\psi^* \frac{\partial \psi}{\partial \theta} - \psi \frac{\partial \psi^*}{\partial \theta} \right) = \mathcal{N}^2 \frac{\nu J_\nu^2(\kappa r)}{r}. \qquad (49)$$

Analogously, $\zeta_r = \zeta_\theta = 0$ and the only non-zero component of the vorticity is

$$\zeta_z = \frac{1}{r} \frac{\partial(r u_\theta)}{\partial r} = \mathcal{N}^2 \frac{\nu}{r} \frac{\mathrm{d}}{\mathrm{d}r} J_\nu^2(\kappa r). \qquad (50)$$

By introducing the vortex radius R as the value at which $u_\theta(r = R) = 0$, and using the relation

$$\int_0^R r\mathrm{d}r J_\nu^2(\kappa r) = \frac{R^2}{2} J_{\nu+1}^2(\kappa R) \qquad (51)$$

(with $J_\nu(\kappa R) = 0$), we can set finally $\mathcal{N} = \sqrt{\frac{2}{R}} \frac{1}{J_{\nu+1}(\kappa R)}$.

Therefore, a reasonable value of the cutoff momentum for single-roton states is

$$q_{\text{max}} \sim \kappa_{\text{max}} \sim \frac{1}{R_{\text{min}}}, \qquad (52)$$

where R_{min} denotes the minimum transverse size of the thinnest vortices that can be established in the superfluid. These are the so-called "vortex filaments", whose transverse size can be obtained from [42] in terms of the particle density n and of the scattering length a. In this case, for a dilute ("almost ideal") Bose condensate, the filament "core" is

$$R_{\text{min}} \equiv r_{\text{core}} \sim \frac{1}{\sqrt{na}}, \qquad (53)$$

so that, for $n = \bar{n}$, we find

$$q_{\text{max}} \sim \frac{1}{r_{\text{core}}} \sim \sqrt{\bar{n}a}. \qquad (54)$$

In this way, assuming for the mass of the fluid constituents the same equation, (28), as established for the quanta of the scalar condensate, one finds

$$\bar{\rho} = m_\Phi \bar{n} \sim \bar{n}^2 a^2, \tag{55}$$

so that replacing relations (54) and (55) by (41) and (42), one gets the order of magnitude estimate

$$\Delta \sim \mu \sim \sqrt{\bar{n}a}. \tag{56}$$

Therefore, by comparing with (21), we have identified the relativistic regime $\Delta = \mu = M_H$ foreseen in the Introduction. In fact, M_H coincides with the energy gap Δ for vortex formation in a superfluid medium possessing the same density and the same type of constituents as the scalar condensate.

More generally, it is interesting to compare the possible regimes of a Bose superfluid, made up of particles with mass m, in terms of the two dimensionless parameters

$$x = \frac{m}{na^2}, \tag{57}$$

and

$$\epsilon = \sqrt{na^3}. \tag{58}$$

In terms of x, using (54) for arbitrary n, (41) and (42) give

$$\Delta \sim \frac{\sqrt{na}}{x} \tag{59}$$

and

$$\mu \sim x\sqrt{na} \sim x^2 \Delta, \tag{60}$$

so that, as anticipated, the Lorentz covariant condition $\Delta \sim \mu$ corresponds to $x \sim 1$.

Notice that the momentum $\delta \sim 1/r_{\mathrm{mfp}} \sim na^2$, related to the transition from the phonon branch to the roton-like excitations of (12), can be used to obtain the sound velocity from the relation

$$c_{\mathrm{s}}\delta \sim \Delta, \tag{61}$$

so that (in units of c)

$$c_{\mathrm{s}} \sim \frac{1}{x\epsilon}. \tag{62}$$

Using (60), the non-relativistic limit, where $\Delta \ll \mu$, is seen to correspond to very large values of x such that also $1 \ll x\epsilon$. In this case, the sound velocity becomes

$$c_{\mathrm{s}} \sim \frac{1}{x\epsilon} \sim \frac{\sqrt{na}}{m} \ll 1, \tag{63}$$

which is the Lee–Yang–Huang result for a dilute hard-sphere Bose gas [43]. On the other hand, a Lorentz covariant form of the massive branch requires a value $x \sim 1$ in (60). This, when replaced in (62), produces the anticipated vastly superluminal sound velocity (29),

$$c_{\mathrm{s}} \sim 1/\epsilon, \tag{64}$$

that diverges in the limit where $\epsilon \to 0$ and the scale $\Delta \sim \mu \sim \sqrt{na} =$ is kept fixed.

The above relations can also be compared with superfluid ^{4}He. Although this is *not* a dilute system (with typical values $a \sim 2.7 \cdot 10^{-8}\,\mathrm{cm}$ and $n \sim 10^{23}\,\mathrm{cm}^{-3}$, one gets $\epsilon \sim 1.4$), we find, nevertheless, a good agreement with our picture. In fact, using the experimental values [44] $(\mu)_{\exp} = 0.16 m_{\mathrm{He}}$ and $(\Delta)_{\exp} = 7.4 \cdot 10^{-4}\,\mathrm{eV}$ in (60) one can obtain an experimental value of x, say

$$x_{\exp} \equiv \sqrt{\frac{(\mu)_{\exp} c^2}{(\Delta)_{\exp}}} \sim 10^6, \tag{65}$$

that, if used in (62), produces a value $c_{\mathrm{s}} \sim 0.7 \cdot 10^{-6}$, in good agreement with the experimental result for the sound velocity $(c_{\mathrm{s}})_{\exp} = 239\,\mathrm{m/s}$ [44]. Finally, the theoretical input prediction from (57)

$$x_{\mathrm{th}} \equiv \frac{m_{\mathrm{He}}}{na^2} \sim 3 \cdot 10^6, \tag{66}$$

is also in fairly good agreement with the experimental result (65), thus confirming the overall consistency of our picture.

4 Summary and outlook

Taking into account the two-valued nature [18,19] of the zero-four-momentum connected propagator in the broken phase, one gets the idea of a true physical medium that contains, for $\mathbf{k} \to 0$, two types of excitations: a massive one, whose energy $E_a(\mathbf{k}) \to M_H$ and that corresponds to the usual Higgs boson field, and a gapless one whose energy $E_b(\mathbf{k}) \to 0$. The overall picture is similar to the coexistence of phonons and rotons in superfluid ^{4}He that, in fact, is usually considered the condensed-matter analogue of the Higgs condensate.

Now, the gapless branch is naturally interpreted in terms of the collective density fluctuations of the system [15–17]. These dominate the physical spectrum for $\mathbf{k} \to 0$ and their wavelengths are larger than r_{mfp}, the mean free path for the condensed quanta.

On the other hand, the continuum limit of quantum field theory corresponds to the ideal case of an incompressible fluid so that the massive energy spectrum $E_a(\mathbf{k})$ extends down to $\mathbf{k} = 0$.

In this paper, following the original Ziman's [35] approach and using the formalism of [2], we have proposed the interpretation of the massive branch as a roton, i.e. as an elementary excitation that, differently from the phonons associated with the irrotational motions of the superfluid, arises in connection with a non-zero ("bulk") vorticity. This interpretation requires the energy spectrum (12) to exhibit values of Δ and μ such that

$$\Delta \sim \mu \sim M_H \sim \sqrt{\bar{n}a}. \tag{67}$$

In turn, this relation depends on the peculiar condition (28),

$$m_\Phi \sim \bar{n}a^2, \tag{68}$$

between the mass m_Φ of the elementary condensing phions, their equilibrium number density $\bar{n}$ and their scattering length a. The relation (68) implies that the phion mean free path $r_{\mathrm{mfp}} \sim 1/(\bar{n}a^2)$ is of the same order as the phion Compton wavelength $1/m_\Phi$ and is naturally found in the weakly first-order scenario of [2], where the broken phase is represented as a dilute Bose condensate for which $\epsilon = \sqrt{\bar{n}a^3} \ll 1$, the continuum limit corresponding to $\epsilon \to 0$ with $\bar{n}a = $ fixed.

Of course, it being usual to consider Lorentz covariance as an exact built-in requirement, an energy spectrum as in (12) with $\Delta = \mu$ may seem more or less trivial. For instance, starting from the original annihilation and creation operators for the elementary quanta of the unphysical empty vacuum state $|0\rangle$, it is easily discovered within the standard covariant generalization of the Bogolubov method [2]. However, our analysis shows that there is an alternative viewpoint. In fact, within quantum hydrodynamics the same result is far from being trivial and is only recovered in the special case (68). In this modified perspective, a Lorentz covariant massive spectrum corresponds (with some approximations and in a certain range of wavelengths) to vortex formation in a superfluid medium possessing a well-defined pattern of scales.

We conclude by observing that the proposed identification of the massive Higgs boson as a roton and of M_H with the energy gap for vortex formation in the superfluid vacuum is not a mere exercise. In fact, in quantum hydrodynamics the single-roton states have a natural cutoff momentum at

$$q_{\max} \sim \sqrt{\bar{n}a} \sim M_H \ll \Lambda. \tag{69}$$

Therefore, accepting our interpretation, the Higgs boson momentum would be physically cut off at values that are much smaller than Λ so that M_H emerges as a naturally intermediate, fixed energy scale associated with the condensed vacuum. In this sense, by treating the Higgs condensate as a real physical medium, one can find a solution of the so-called *naturalness* problem without any artificial fine-tuning of the basic parameters. Such finite-tuning problems, instead, appear in the standard approach where the massive Higgs boson is regarded as an ordinary elementary particle propagating in the vacuum and its maximum momentum is identified with the ultimate ultraviolet cutoff Λ of the theory.

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